

ON EXTREMISERS TO A BILINEAR STRICHARTZ INEQUALITY

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ABSTRACT. In this note, we show that a pair of Gaussian functions are extremisers to a bilinear Strichartz inequality, and unique up to the symmetry group of the inequality.

1. INTRODUCTION

We consider the free Schrödinger equation

$$(1) \quad i\partial_t u + \Delta u = 0,$$

with initial data $u(0, x) = f(x)$ where $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ is a complex-valued function and $d \geq 1$. We denote the solution u by using the Schrödinger evolution operator $e^{it\Delta}$:

$$(2) \quad u(t, x) := e^{it\Delta} f(x) := \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|^2} \widehat{f}(\xi) d\xi,$$

where \widehat{f} is the spatial Fourier transform of f defined via

$$(3) \quad \widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(x) dx,$$

where $x \cdot \xi$ (abbr. $x\xi$) denotes the Euclidean inner product of x and ξ in the spatial space \mathbb{R}^d . When $f \in L^2(\mathbb{R}^d)$, the solution $e^{it\Delta} f$ enjoys a space-time estimate

$$(4) \quad \|e^{it\Delta} f\|_{L_{t,x}^{2+\frac{4}{d}}(\mathbb{R} \times \mathbb{R}^d)} \leq C_d \|f\|_{L_x^2(\mathbb{R}^d)},$$

for some $C_d > 0$, see e.g. [8] or [12]. Define

$$C_d := \sup \left\{ \frac{\|e^{it\Delta} f\|_{L_{t,x}^{2+\frac{4}{d}}(\mathbb{R} \times \mathbb{R}^d)}}{\|f\|_{L_x^2(\mathbb{R}^d)}} : f \in L^2(\mathbb{R}^d) f \neq 0 \right\}.$$

Several authors have investigated the extremal problem for (4), which asks whether there is an extremal function $f \in L^2(\mathbb{R}^d)$ such that

$$\|e^{it\Delta} f\|_{L_{t,x}^{2+\frac{4}{d}}(\mathbb{R} \times \mathbb{R}^d)} = C_d \|f\|_{L^2(\mathbb{R}^d)},$$

and what properties the extremal functions have. More precisely, Kunze [10] treated the $d = 1$ case and showed that extremisers exist by an elaborate concentration-compactness method; when $d = 1, 2$, Foschi [7] explicitly determined the best constants and showed that the extremisers are Gaussians, and they are unique up to the symmetry of the Strichartz inequality. Hundertmark and Zharnitsky [9] independently obtained this result. Carneiro [3] consider similar extremal questions for some Strichartz-type inequality. In [1], by using the method of heat-flow, Bennett, Bez, Carbery and Hundertmark offered a new proof to determine the best constants and the explicit form of extremisers for the symmetric inequality (4) when $d = 1, 2$. When $d \geq 3$, only the existence of extremisers for the symmetric Strichartz inequality has been shown, see e.g. [11]. Similar extremal questions have been treated for the Fourier restriction inequality for the hyper-surfaces in the Euclidean spaces such as the sphere in [5, 6], and the Strichartz inequality for the wave equation in [2, 7].

In this paper, we specify the dimension $d = 2$ and consider the extremal problem for a bilinear Strichartz inequality for the Schrödinger operator,

$$(5) \quad \|e^{it\Delta} f e^{it\Delta} g\|_{L^2_{t,x}(\mathbf{R} \times \mathbf{R}^2)} \leq \mathbf{B} \|f\|_{L^2(\mathbf{R}^2)} \|g\|_{L^2(\mathbf{R}^2)}.$$

where \mathbf{B} is the optimal constant defined by

$$\mathbf{B} := \sup_{f \neq 0, g \neq 0} \frac{\|e^{it\Delta} f e^{it\Delta} g\|_{L^2_{t,x}(\mathbf{R} \times \mathbf{R}^2)}}{\|f\|_{L^2(\mathbf{R}^2)} \|g\|_{L^2(\mathbf{R}^2)}}.$$

We define an extremiser or an extremal function to (5) is a pair of functions $(f, g) \in L^2 \times L^2$ such that

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2_{t,x}(\mathbf{R} \times \mathbf{R}^2)} = \mathbf{B} \|f\|_{L^2(\mathbf{R}^2)} \|g\|_{L^2(\mathbf{R}^2)}.$$

It is well known that the linear Strichartz inequality (4) is invariant under the following symmetry group G generated by

- Translation. $e^{it\Delta} f(x) \rightarrow e^{i(t-t_0)\Delta} f(x-x_0)$ for any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^2$.
- Scaling. $f(x) \rightarrow \lambda^2 f(\lambda x)$ for any $\lambda > 0$.
- Galilean transform. $e^{it\Delta} f(x) \rightarrow e^{ix \cdot \xi_0 + it|\xi_0|^2} f(x + 2t\xi_0)$ for any $\xi_0 \in \mathbf{R}^2$.
- Phase transition. $e^{it\Delta} f(x) \rightarrow \alpha e^{i\theta_0} e^{it\Delta} f(x)$ for any $\theta_0 \in \mathbf{R}$ and $\alpha \in \mathbf{C}$.
- Space rotation. $e^{it\Delta} f(x) \rightarrow e^{it\Delta} f(Rx)$ for any $R \in SO(2)$.

For the extremal problem for the linear Strichartz inequality, it is true that the symmetry group G will change an extremal function to another; so an extremal function f will generate a family of extremal functions under the action of G . This family of functions is called the orbit of f . Because the

inequality (5) is invariant under the symmetry in G , it is also the case for the extremal function. Now we state the following result.

Theorem 1.1. *The pair of Gaussian functions*

$$(f, g) = (\exp(-|x|^2), \exp(-|x|^2))$$

is an extremiser to the bilinear Strichartz inequality (5), and $B = 2\pi^2$. Moreover, the set of extremisers for which (5) holds coincides with the orbit of

$$(f, g) = (\exp(A|x|^2 + b \cdot x + C), \exp(A|x|^2 + b \cdot x + C)),$$

under the action of the symmetry group G , where $A, C \in \mathbf{C}$ and $b \in \mathbf{C}^2$.

To prove it, we follow closely Foschi's simple argument in [7], where it is shown that Gaussians are the only extremisers to the linear Strichartz inequality

$$(6) \quad \|e^{it\Delta} f\|_{L^4_{t,x}(\mathbf{R} \times \mathbf{R}^2)} \leq C \|f\|_{L^2(\mathbf{R}^2)}$$

up to the symmetry in G .

Remark 1.2. An analogous theorem to Theorem 1.1 can be established for a trilinear Strichartz inequality in the one-dimensional case,

$$(7) \quad \|e^{it\Delta} f e^{it\Delta} g e^{it\Delta} h\|_{L^2_{t,x}(\mathbf{R} \times \mathbf{R})} \leq C \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.$$

Remark 1.3. Recently it comes to our attention that M. Charalambides [4] systematically investigated the question of characterizing functions f, g and h such that the Cauchy-Pexider functional equation $f(x)g(y) = h(x+y)$ with x, y on some hyper-surface in R^{d+1} . The solutions to such functional equation are uniquely determined to be exponential affine functions. This is closely connected to Theorem 1.1 because the functional equation characterizing the sharpness of the bilinear Strichartz inequality (5) is in the same form, see Section 3.

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2. NOTATION AND PRELIMINARY

We begin with some notation. Define the Fourier transform,

$$\widehat{f}(\xi) = \int_{\mathbf{R}^3} e^{ix \cdot \xi} f(x) dx, \quad \xi \in \mathbf{R}^3.$$

The inverse of the Fourier transform,

$$f(x) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} e^{-ix \cdot \xi} \widehat{f}(\xi) d\xi, \quad x \in \mathbf{R}^3.$$

The Plancherel theorem states that

$$\|f\|_{L^2(\mathbf{R}^3)} = \frac{1}{(2\pi)^{3/2}} \|\widehat{f}\|_{L^2(\mathbf{R}^3)}.$$

Moreover the Parseval identity states that

$$\int_{\mathbf{R}^3} f(x)g(x)dx = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi.$$

Let σ be the endowed measure on the paraboloid $P := \{(\tau, \xi) : \tau = |\xi|^2\}$ in \mathbf{R}^3 , defined to be the pullback of the Lebesgue measure in $\mathbf{R} \times \mathbf{R}^2$ under the projection map: $(|\xi|^2, \xi) \mapsto \xi$. Then it follows that

$$e^{it\Delta}f(x) = \int_{\mathbf{R}^2} e^{ix \cdot \xi + it|\xi|^2} f(\xi) d\xi = \widehat{f\sigma}(t, x).$$

We define the convolution of f and g ,

$$f * g(x) = \int_{\mathbf{R}^3} f(x - y)g(y)dy,$$

and record a useful identity about convolution under the action of Fourier transform

$$(8) \quad \widehat{f * g} = \widehat{f}\widehat{g} \text{ and } \widehat{\widehat{f}\widehat{g}} = (2\pi)^3 f * g.$$

We also record several lemmas from Foschi [7]. The first is the well-known Cauchy-Schwarz inequality, see e.g. [7, Lemma 1.8].

Lemma 2.1. *Let $\langle \cdot, \cdot \rangle$ be a (complex) inner product on a vector space V and let $u, v \in V$ be two non-zero vectors. The Cauchy-Schwarz inequality states that*

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle;$$

Moreover, the equality holds if and only if $u = \alpha v$ for some scalar $\alpha \in \mathbf{C}$.

The second lemma shows that the convolution of the surface measure σ is constant on its corresponding support $\Omega := \{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^2 : 2\tau \geq |\xi|^2\}$, see e.g. [7, Lemma 3.2] .

Lemma 2.2.

$$(9) \quad \text{If } x \in \Omega, \quad \sigma * \sigma(x) = \pi/2.$$

Finally we cite a lemma [7, Proposition 7.15] on characterizing the following functional inequality,

$$(10) \quad f(x)f(y) = F(|x|^2 + |y|^2, x + y), \text{ for almost everywhere } (x, y) \in \mathbf{R}^2 \times \mathbf{R}^2.$$

Lemma 2.3. *If $f : \mathbf{R}^2 \rightarrow \mathbf{C}$ and $F : \Omega \rightarrow \mathbf{C}$ are nontrivial locally integrable functions which satisfy the functional equation (10), then there exists constants $A \in \mathbf{C}$, $b \in \mathbf{C}^2$ and $C \in \mathbf{C}$ such that*

$$f(x) = \exp(A|x|^2 + b \cdot x + C), \quad F(t, x) = \exp(At + b \cdot x + 2C)$$

for almost all $(t, x) \in \Omega$.

3. THE PROOF

Before we prove Theorem 1.1, we recall Foschi's argument in [7]. Foschi establishes the inequality (6) with an explicit constant by the Cauchy-Schwarz inequality. The only place where an inequality sign occurs is due to the Cauchy-Schwarz inequality. Then the question reduces to what functions make the Cauchy-Schwarz inequality sharp in the sense that the inequality becomes equal. This yields a functional equation, whose solutions uniquely determine extremisers. We will apply this idea.

Proof of Theorem 1.1. By the Plancherel theorem,

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2} = \|\widehat{f\sigma g\sigma}\|_{L^2} = \frac{1}{(2\pi)^{3/2}} \|\widehat{f\sigma g\sigma}\|_{L^2}.$$

Then by (8), the above

$$= (2\pi)^{3/2} \|f\sigma * g\sigma\|_{L^2}.$$

Denote μ by the measure $\sigma * \sigma$ defined via

$$(11) \quad f\sigma * g\sigma(x) = \int f(x-y)g(y)d\mu(y).$$

By the Cauchy-Schwarz inequality, this representation quickly yields

$$(12) \quad (f\sigma * g\sigma)^2 \leq (f^2\sigma * g^2\sigma)(\sigma * \sigma).$$

Then

$$(13) \quad \begin{aligned} \|f\sigma * g\sigma\|_{L^2}^2 &= \int (f\sigma * g\sigma)^2 dx \\ &\leq \int (f^2\sigma * g^2\sigma)(\sigma * \sigma) dx \\ &= \frac{\pi}{2} \int f^2\sigma * g^2\sigma dx = \frac{\pi}{2} \|f\|_{L_\sigma^2}^2 \|g\|_{L_\sigma^2}^2, \end{aligned}$$

by Lemma 2.2 that

$$(14) \quad \sigma * \sigma(x) = \frac{\pi}{2}, \text{ for all } x \in \Omega.$$

To conclude so far, we have established the bilinear Strichartz inequality (5) with an explicit constant $2\pi^2$.

If the Cauchy-Schwarz inequality used in (13) is sharp in the sense that an equal sign occurs in (12), then all the inequalities in (13) become equal; that is to say

$$\|f\sigma * g\sigma\|_{L^2} = \sqrt{\frac{\pi}{2}} \|f\|_{L_\sigma^2} \|g\|_{L_\sigma^2}.$$

An examination of the sharpness of the Cauchy-Schwarz inequality shows that

$$(15) \quad f\sigma * g\sigma(x) = \alpha\sigma * \sigma(x), \text{ for a.e. } x \in \mathbf{R}^3 \text{ and } \alpha \in \mathbf{R}.$$

This further shows that

$$(16) \quad f(x)g(y) = F(|x|^2 + |y|^2, x + y) \text{ for a.e. } x \times y \in \mathbf{R}^2 \times \mathbf{R}^2.$$

for some measurable function F .

Since $f \neq 0$, without loss of generality, we may assume that $f(0) \neq 0$. Let $x = 0$ and $y = 0$ in (16), respectively,

$$f(0)g(x) = f(x)g(0) \text{ for a.e. } x \in \mathbf{R}^2.$$

Then

$$g(x) = \frac{g(0)}{f(0)}f(x).$$

In view of this, we may assume that $f = g$. This assumption reduces (16) to

$$(17) \quad f(x)f(y) = F(|x|^2 + |y|^2, x + y) \text{ for a.e. } x \times y \in \mathbf{R}^2 \times \mathbf{R}^2.$$

Then by Lemma 2.3, there exists constants $A \in \mathbf{C}$, $b \in \mathbf{C}^2$, and $C \in \mathbf{C}$ such that

$$(18) \quad f(x) = \exp(A|x|^2 + b \cdot x + C), \quad F(t, x) = \exp(At + b \cdot x + 2C)$$

for almost everywhere $(t, x) \in P$. Thus we have shown that the extremisers to (5) are Gaussian functions, which are also unique up to the symmetry group G . This completes the proof of Theorem 1.1. \square

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